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THE CALCULATION OF STRONGLY NON-LINEAR SYSTEMS CLOSE TO VIBRATION IMPACT SYSTEMS *

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A method of constructing periodic solutions in non-linear dynamic systems with a finite number of degrees of freedom, which enables a solution to be found in the form of a series, from periodic piecewise smooth functions of a fairly simple form is proposed. The approximate solution, which is a segment of the series, corresponds to replacing the initial system by a certain equivalent vibration impact system. As shown below, such an approach is particularly effective in the least favourable cases for quasiharmonic (quasilinear) analysis. In combination with averaging methods this approach can also be used to study more complicated modes of motion. Some examples which are of independnet interest are discussed.

1. First we consider a conservative system with one degree of freedom whose motion is described by the equation

$$x^{\cdot\cdot} + f(x) = 0, \quad x \in R_1 \tag{1.1}$$

Here $f(x)$ is the odd analytic function which satisfies the condition $xf(x) \geqslant 0$, where equality occurs only at one point $x = 0$ (the system with a unique state of equilibrium); a dot denotes differentiation with respect to time t.

The group properties of Eq.(1.1) enable us, without loss of generality, to consider this equation under the following initial conditions:

$$t = 0, \; x = 0, \; x^{\cdot} = v \tag{1.2}$$

Let $P(\varphi)$ be a sawtooth periodic piecewise smooth function with unit amplitude

$$P(\varphi) = \frac{2}{\pi} \arcsin\left(\sin \frac{\pi\varphi}{2}\right), \quad P(\varphi + 4) = P(\varphi)$$

A graph of such a function consists of joined straight-line segments, and it can therefore be calculated using simple arithmetic operations.

The relations

$$P'^2(\varphi) = 1, \; P''(\varphi) = 2 \sum_{k=-\infty}^{\infty} [\delta(\varphi + 1 - 4k) - \delta(\varphi - 1 - 4k)], \tag{1.3}$$
$$-\infty < \varphi < \infty$$

hold within the scope of the theory of generalized functions.

We shall seek the solution of the Cauchy problem (1.1), (1.2) in the form

$$x = \psi + X(\psi), \; \psi = AP(\varphi), \; \varphi = vt/A \tag{1.4}$$

With respect to the variable t, the solution period T is determined by the expression $T = 4A/v$; for example, the parameter A and the function X are to be determined.

On differentiating (1.4) twice with respect to t, and considering (1.3), we obtain

$$x^{\cdot\cdot} = v^2X'' + (v^2/A)\,(1 + X')\,P'' \tag{1.5}$$

The acceleration in the system discussed should be limited, and for this reason we eliminate the second term in (1.5) by putting

$$X'\,|_{\psi=A} = -1 \tag{1.6}$$

Because the function $\psi = AP(\varphi)$ is periodic and $X'(\psi)$ is even, equality (1.6) is satisfied at all points $\varphi = \pm 1 + 4k, k = 0, \pm 1, \pm 2, \ldots$ Eq.(1.6) serves to determine the parameter A.

On substituting (1.4) into the initial Eq.(1.1), and considering (1.5), (1.6), we obtain the equation in $X(\psi)$:

$$v^2X'' = -f(\psi + X) = -f(\psi) - f'(\psi)X - \frac{1}{2}f''(\psi)X^2 - \ldots \tag{1.7}$$

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The corresponding initial conditions

$$\psi = 0,\ X = 0,\ X' = 0 \tag{1.8}$$

follow from relations (1.2).

We shall seek the solution of problem (1.7), (1.8) in the form of the series of successive approximations

$$X = X^{(1)} + X^{(2)} + X^{(3)} + \ldots, A = A^{(1)} + A^{(2)} + A^{(3)} + \ldots \tag{1.9}$$

On substituting the first series of (1.9) into (1.7), we obtain the sequence of equations

$$X^{(1)''} = -v^{-2}f(\psi),\ X^{(2)''} = -v^{-2}f'(\psi)\ X^{(1)}$$
$$X^{(3)''} = -v^{-2}\left[f'(\psi)X^{(2)} + \frac{1}{2}f''(\psi)X^{(1)2}\right], \ldots$$

Hence, considering the initial conditions (1.8), we arrive at relations which by calculating the quadratures, enable us to determine the terms of the first expansion (1.9). On integrating by parts the terms which contain the derivatives of the function $f(x)$, we have

$$X^{(1)} = -v^{-2}\int_0^{\psi}\int_0^{\psi} f(\psi)\,d\psi\,d\psi \tag{1.10}$$
$$X^{(2)} = -v^{-2}\Big[\int_0^{\psi} f(\psi)X^{(1)}\,d\psi - \int_0^{\psi}\int_0^{\psi} f(\psi)X^{(1)'}\,d\psi\,d\psi\Big]$$
$$X^{(3)} = -v^{-2}\Big\{\frac{1}{2}f(\psi)X^{(1)2} + \int_0^{\psi} f(\psi)(X^{(2)} - 2X^{(1)}X^{(1)'})\,d\psi + \int_0^{\psi}\int_0^{\psi} f(\psi)[(X^{(1)}X^{(1)'})' - X^{(2)'}]\,d\psi\,d\psi\Big\}, \ldots$$

On substituting the expansions (1.9) into (1.6), and expanding the derivatives $X^{(i)'}$ ($i = 1, 2, \ldots$) in power series in the vicinity of the point $\psi = A^{(1)}$, we obtain a chain of equations for determining the quantities $A^{(j)}$ ($j = 1, 2, \ldots$) :

$$X^{(1)'}\big|_{\psi=A^{(1)}} = -1,\quad A^{(2)} = -\frac{X^{(2)'}}{X^{(1)''}}\bigg|_{\psi=A^{(1)}} \tag{1.11}$$
$$A^{(3)} = -\frac{1}{X^{(1)''}}\left(\frac{1}{2}X^{(1)'''}A^{(2)2} + X^{(2)''}A^{(2)} + X^{(3)'}\right)\big|_{\psi=A^{(1)}}, \ldots$$

Bearing in mind (1.10), we can reduce the first of these equations to the form

$$\int_0^{A^{(1)}} f(\psi)\,d\psi = v^2 \tag{1.12}$$

Thus, the first term of the second expansion in (1.9) equals the oscillation amplitude which occurs if the given initial energy is doubled.

Note that relations (1.10) do not contain derivatives of the function $f(\psi)$, and constructing an iteration procedure based on a simpler scheme generally enables us to avoid differentiation of this function, thereby removing the condition of analyticity imposed on it.

For example, let

$$X^{(0)} \equiv 0,\quad X^{(i)} = -v^{-2}\int_0^{\psi}\int_0^{\psi} f(\psi + X^{(i-1)})\,d\psi\,d\psi,\quad i = 1, 2, \ldots$$

Then the function $X^{(N)}$ for sufficiently large N will be an approximate solution of Eq.(1.7). As regards $X^{(1)}$ we have expression (1.10); however, in the higher approximations the calculation of the quadratures appears to be much more complex than in (1.10), and the result is less clear.

Below we give examples of the singularities of the approach described.

Example 1. Let $f(x) \equiv x^n$, where n is an odd number. In this case for arbitrary n, problem (1.1), (1.2) can be solved with special functions of a quite complex form. We can arrive at simpler solutions for the analysis (even if approximate ones) in another way. Thus for small n, the quasiharmonic approach often gives a quite acceptable result which, by one means or another, includes a certain linearization of the initial system. However, if n is large, this approach appears to be unnatural.

From this point of view, the limit case $n = \infty$ appears to be fundamentally complex, since to describe the impacts at the ends of the segment $-1 \leqslant x \leqslant 1$, strictly speaking, an infinite number of quasiharmonic approximations are required.

At the same time this case is very simple because in the intervals between the impacts uniform motion with respect to inertia occurs, and in the wider class of functions the solution of the problem is

$$x = P(\varphi),\ \varphi = vt \tag{1.13}$$

We also note that, if relations (1.13) are regarded as a change of variable in (1.1), then in the equation in the new variable φ the singular functions which correspond to the impact interaction disappear, and the equation takes the form $\varphi'' = 0,\ -\infty < \varphi < \infty$. The idea of using special functions (including the sawtooth) in calculating vibration impact systems is discussed in /1/.

The above relations enable us to obtain an approximate solution for large but finite values of n. On performing the integration in (1.10), we find the solution of the initial problem in the form of a power series,

$$x = \psi - \frac{\psi^{n+2}}{v^2(n+1)(n+2)} + \frac{n\psi^{2n+3}}{2v^4(n+1)^2(n+2)(2n+3)} - \frac{n}{6v^6(n+1)^3(n+2)(3n+4)}\left(\frac{n-1}{n+2} + \frac{n}{2n+3}\right)\psi^{3n+4} + \ldots \tag{1.14}$$
$$\psi = AP(vt/A)$$

Regarding the parameter A in (1.11), we have

$$A = A^{(1)} + \frac{n}{2v^2(n+1)^2(n+2)}A^{(1)\,n+2} + \left(1 - \frac{n-1}{3n} - \frac{n+2}{6n+9} - \frac{n}{4n+4}\right)\frac{n^2A^{(1)\,2n+3}}{2v^4(n+1)^3(n+2)^2} + \ldots, \quad A^{(1)} = [(n+1)v^2]^{1/(n+1)} \tag{1.15}$$

The series (1.14) and (1.15) are asymptotic: as $n\to\infty$ we obtain, as should be the case, relations (1.13):

$$A \to 1,\ X = x - A\psi \to 0;\ x \to P(vt)$$

In order to establish the convergence, we analyse the least favourable case $n = 1$ (an harmonic oscillator). For $n = 1$, expressions (1.14) and (1.15) take the form

$$x = v\left(\frac{\psi}{v} - \frac{1}{3!}\frac{\psi^3}{v^3} + \frac{1}{5!}\frac{\psi^5}{v^5} - \frac{1}{7!}\frac{\psi^7}{v^7} + \ldots\right) = v\sin\frac{\psi}{v}$$
$$A = \sqrt{2}v\left(1 + \frac{1}{12} + \frac{3}{160} + \ldots\right) = \frac{\pi}{2}v$$

thus restoring the solution of a linear equation represented in the form

$$x = v\sin\tau,\ \tau = {}^1/_2\,\pi P(2t/\pi)$$

Unlike the usual form of notation of a solution $(x = v\sin t)$, the approximation of the function $\sin\tau$ by a segment of the power series does not deprive it of the property of periodicity, transferred to the 'oscillating' time $\tau, |\tau| \leqslant \pi/2$; on the other hand, any finite segment of the series deprives the approximated function of the property of being smooth. The kinks in the corresponding curve are smoothed with additional new terms of the series, and disappear for infinitely large numbers of terms only. At the kinks, the velocity function has discontinuities of the first kind which correspond to some fictitious impacts in the system. However, for large n the approximation of rapid velocity jumps by discontinuities, and the impulses of force by instantaneous impulses is essential. Regarding a linear system, the expansion of the sine with respect to the 'saw' in a power series is obviously insufficient to the same degree as is the expansion of the 'saw' in a Fourier series with respect to sines in the case of a vibration-impact system $(n = \infty)$.

Example 2. Consider the solution for an oscillator with a characteristic of the form $f(x) = \operatorname{sh} x$. If in the process of integrating the equations we digress from the initial conditions for the derivatives $X^{(i)\prime}(i = 1,2,\ldots)$, the single-parameter family of solutions of the initial equation can finally be obtained in the form

$$x = \psi - \frac{1}{V^2}\operatorname{sh}\psi + \frac{1}{8V^4}\operatorname{sh}2\psi - \frac{1}{48V^6}(\operatorname{sh}3\psi - 15\operatorname{sh}\psi) + \ldots \tag{1.16}$$
$$\psi = AP\left(\frac{Vt}{A}\right), \quad A = \ln(V^2 + \sqrt{V^4 - 1}) + \frac{2V^4 - 1}{4V^2\sqrt{V^4 - 1}} + \ldots$$

The parameter V is connected with the initial velocity by the relation

$$v = V - \frac{1}{V} + \frac{1}{4V^3} + \frac{1}{4V^5} + \ldots$$

For sufficiently large v, the quantity V can be replaced approximately by the velocity v, and as $v\to\infty$ the asymptotic behaviour of (1.16) can be expressed as

$$x \sim 2\ln vP\left(\frac{vt}{2\ln v}\right) \tag{1.17}$$

Thus, the system suffers impacts at the infinitely removed points $x = \pm\infty$ reached in an infinitely short time.

2. Consider a conservative system with $k + 1$ degrees of freedom. To simplify the notation, we specially separate one of the coordinates, (x), and write the eqution of motion in the form

$$x^{\cdot\cdot} + f(x, y) = 0,\ y^{\cdot\cdot} + g(x, y) = 0 \tag{2.1}$$

where y denotes the aggregate k of the quantities $y_1, y_2, \ldots, y_k$; $g(x, y)$ is the aggregate of the functions $g_i = g_i(x, y_1, y_2, \ldots, y_k)$ $(i = 1, 2, \ldots, k)$, which, together with $f(x, y)$, are assumed to be analytic.

Let the potential function $U(x, y)$ corresponding to (2.1) satisfy the symmetry condition $U(x, y) = U(-x, -y)$, and in addition let the set $\Gamma = \{(x, y): U(x, y) + h = 0\}$ (h is the system's energy) represent a closed connected hypersurface in R^{k+1} on which the gradient of the function $U(x, y)$ vanishes nowhere (there are no 'stagnation' points). This hypersurface limits the configuration space of the motion which contains a state of equilibrium, in this case unique, i.e. the origin of coordinates.

We shall seek the periodic solution such that at some instants of time all coordinates vanish simultaneously, and at some other instants all derivatives $x^{\cdot}, y^{\cdot}$ are zeros (oscillation in unison). The corresponding trajectory in configuration space has the form of a segment of a line with its ends onthe hypersurface Γ, which passes through the origin. The question of the existence of such solutions is discussed in /2/; a construction method exists for weakly curved trajectories in configuration space (see /3/).

We shall write the initial condition corresponding to the solution discussed in the form

$$t = 0,\ x = 0,\ x^{\cdot} = v,\ y = 0 \tag{2.2}$$

The relevant initial values of the velocitires $y^{\cdot}$ are determined when constructing the solution which in this case we seek in the form

$$x = \psi + X(\psi),\ y = Y(\psi);\ \psi = AP(vt / A) \tag{2.3}$$

where $Y(\psi) = \{Y_1(\psi), Y_2(\psi), \ldots, Y_k(\psi)\}$. For $g(x, y) \equiv 0$ we have $y \equiv 0$, and the trajectory of the solution desired lies on the x-axis. The choice of this or any other axis for the initial approximation is made taking into account sufficient information, chiefly on the system's properties of symmetry.

Let us substitute relations (2.3) into (2.1). Under the conditions

$$\psi = A,\ X' = -1,\ Y' = 0 \tag{2.4}$$

there are no δ-functions in the expressions for the accelerations $x^{\cdot\cdot}$ and $y^{\cdot\cdot}$, and the equations in X and Y have the form

$$v^2X'' + f(\psi + X, Y) = 0,\ v^2Y'' + g(\psi + X, Y) = 0 \tag{2.5}$$

From the conditions imposed on x and $x^{\cdot}$ in (2.2), we have the initial conditions for th function X,

$$\psi = 0,\ X = 0,\ X' = 0 \tag{2.6}$$

The last relations in (2.2) and (2.4) yield the boundary conditions

$$\psi = 0,\ Y = 0;\ \psi = A,\ Y' = 0 \tag{2.7}$$

As was discussed in Section 1, the parameter A is determined by the condition regarding X' in (2.4).

On expanding the functions f and g in power series in the neighbourhood of the point $(\psi, 0)$, and setting

$$X = X^{(1)} + X^{(2)} + X^{(3)} + \ldots,\quad Y = Y^{(1)} + Y^{(2)} + Y^{(3)} + \ldots;$$
$$Y^{(i)} = \{Y_1^{(i)}, \ldots, Y_k^{(i)}\}$$

we obtain the following systems of equations:

$$\begin{aligned}
&v^2X^{(1)''} = -f(\psi, 0),\ v^2Y^{(1)''} = -g(\psi, 0)\\
&v^2X^{(2)''} = -f_x'X^{(1)} - f_y'Y^{(1)},\quad v^2Y^{(2)''} = -g_x'X^{(1)} - g_y'Y^{(1)}\\
&v^2X^{(3)''} = -f_x'X^{(2)} - f_y'Y^{(2)} - \frac{1}{2}f_{xx}''X^{(1)2} - f_{xy}''X^{(1)}Y^{(1)} -\\
&\quad \frac{1}{2}\left(Y^{(1)}\frac{\partial}{\partial y}\right)^2 f\\
&v^2Y^{(3)''} = -g_x'X^{(2)} - g_y'Y^{(2)} - \frac{1}{2}g_{xx}''X^{(1)2} - g_{xy}''X^{(1)}Y^{(1)} - \frac{1}{2}\left(Y^{(1)}\frac{\partial}{\partial y}\right)^2 g, \ldots
\end{aligned}$$

The symbol f_y' denotes the vector of derivatives $\partial f/\partial y_i$ $(i = 1, 2, \ldots, k)$, and g_y' the square matrix $\| \partial g_j/\partial y_i \|$; all the derivatives of the functions f and g are calculated at the point $(\psi, 0)$.

Separately, each of the functions $X^{(1)}, X^{(2)}, \ldots$ and $Y^{(1)}, Y^{(2)}, \ldots$ should satisfy the initial and the boundary conditions (2.6) and (2.7), respectively.

As an example, let us consider the solution for the following system with two degrees c freedom:

$$x^{\cdot\cdot} + \mathrm{sh}\, x + \gamma\, \mathrm{sh}\,(x - y) = 0,\quad y^{\cdot\cdot} + \mathrm{sh}\, y + \gamma\, \mathrm{sh}\,(y - x) = 0$$

In the main approximation the solution has the form

$$x = \psi + \frac{1+\gamma}{v^2}(\psi - \mathrm{sh}\,\psi),\qquad y = \frac{\gamma}{v^2}(\mathrm{sh}\,\psi - \psi\,\mathrm{ch}\,A)$$

$$\psi = AP\left(\frac{vt}{A}\right),\qquad \mathrm{ch}\,A = 1 + \frac{v^2}{1+\gamma}$$

As $v \to \infty$, the relations

$$x \sim 2\ln vP(\varphi),\qquad y \sim -\frac{\gamma}{1+\gamma}x,\qquad \varphi = \frac{vt}{2\ln v}$$

become valid.

Since the system is symmetric, another solution of this kind can be obtained from the given solution by a simple change $x \rightleftarrows y$.

3. Consider a non-autonomous system with k degrees of freedom. Retaining the notation of Section 2, we write the equations of motion in the vector form:

$$y^{\cdot\cdot} + g(y) = q(t);\; q(t) = \{q_1(t), q_2(t), \ldots, q_k(t)\} \tag{3.1}$$

where $q(t)$ is an aggregate of periodic functions with period $4\tau_0$: $q(t + 4\tau_0) = q(t)$; $q(0) = 0$. Let us intorduce the "oscillating time"

$$\tau = \tau_0 P(t \quad \tau_0)$$

Then, since the function $q(t)$ is periodic, we have the equality

$$q(t) = q(\tau),\quad -\infty < t < \infty$$

We shall seek the perodic solution of (3.1) in the form

$$y = A\tau - Y(\tau);\; Y'|_{\tau=\tau_0} = -A,\; A = \{A_1, A_2, \ldots, A_k\} \tag{3.2}$$

where A are the constants which are to be determined. We obtain

$$Y'' + g(A\tau + Y) = q(\tau) \tag{3.3}$$

Putting $Y = Y^{(1)} + Y^{(2)} + \ldots$, we arrive at a sequence of equations

$$Y^{(1)''} = q(\tau) - g(A\tau),\; Y^{(2)''} = -g_y'Y^{(1)}, \ldots \tag{3.4}$$

where the derivatives g_y' are computed for $y = A\tau$. In integrating these equations we choose the arbitrary constants so that the functions $Y^{(i)}$ $(i = 1, 2, \ldots)$ do not contain a term linear in τ (it is allowed for, by the first expression in (3.2)), and so that they vanish for $\tau = 0$. On substituting the function Y thus obtained into the second relation of (3.2), we obtain a system of equations in the quantities A_j $(j = 1, 2, \ldots, k)$.

Example. Consider the system

$$y_1^{\cdot\cdot} + g_1(y_1, y_2) = Q\tau,\; y_2^{\cdot\cdot} + g_2(y_1, y_2) = 0$$
$$g_1(y_1, y_2) \equiv g_2(y_2, y_1) \equiv y_1^n + \gamma(y_1 - y_2)^n$$

We find a solution of the type discussed in the form

$$y_1 = A_1\tau + \frac{Q}{6}\tau^3 - \frac{g_1(A_1, A_2)}{(n+1)(n+2)}\tau^{n+2} + \cdots$$
$$y_2 = A_2\tau - \frac{g_1(A_2, A_1)}{(n+1)(n+2)}\tau^{n+2} + \cdots$$

For A_1 and A_2 we obtain the equations

$$A_1 = -\frac{Q}{2}\tau_0^2 + g_1(A_1, A_2)\frac{\tau_0^{n+1}}{n+1} + \cdots,\qquad A_2 = g_1(A_2, A_1)\frac{\tau_0^{n+1}}{n+1} + \cdots \tag{3.5}$$

One of the solutions can be found in the form of the power series

$$A_1 = -\frac{Q}{2}\tau_0^2 - \frac{Q^n}{2^n}\frac{1+\gamma}{1+n}\tau_0^{3n-1} + \cdots,\qquad A_2 = \gamma\frac{Q^n}{2^n}\frac{\tau_0^{3n-1}}{n+1} + \cdots$$

Notice that for $n = 1$ (when the system is linear) the written part of Eq. (3.5) proves to be linear and yields the following expression for the resonance quarter-periods:

$$\tau_0^2 = 2\,(1 + \gamma \mp \gamma)^{-1}$$

(the exact expression differs by a factor of $\pi^2/8$).

4. Let us examine the case of parametric action. Suppose that the equation of motion

has the form

$$y^{\cdot\cdot} + [M - Q(t)]\, g(y) + f(y) = 0 \tag{4.1}$$

Here y, as before, is a vector of dimensions k; M and $Q(t)$ are matrices of the k-th order, the elements of the latter being the π-periodic even functions; $g(y)$ and $f(y)$ are the k-dimensional vector-functions which satisfy the symmetry condition: $g(y) = -g(-y)$, $f(x) = -f(-x)$.

We shall seek the odd 2π-periodic solution in the form

$$y = A\tau + Y(\tau);\quad \tau = \frac{\pi}{2} P\left(\frac{2t}{\pi}\right)$$

Under the condition

$$\tau = \pi/2,\ Y' = -A \tag{4.2}$$

we have

$$Y'' = -[M - Q(\tau)]\ g(A\tau + Y) - f(A\tau + Y)$$
$$Y(0) = 0,\ Y'(0) = 0;\ Q(\tau) \equiv Q(t)$$

These relations enable us to write, in the main approximation, the solution of the initial equation in the form

$$y = A\tau - \int_0^\tau\int_0^\tau [(M - Q)\, g(A\tau) + f(A\tau)]\, d\tau\, d\tau \tag{4.3}$$

with the relation (4.2) yielding

$$\int_0^{\pi/2} [(M - Q)\, g(A\tau) + f(A\tau)]\, d\tau = A \tag{4.4}$$

Example. Consider the following system with one degree of freedom:

$$M = a = \text{const},\ Q(t) = 2q\cos 2t,\ g(x) \equiv x,\ f(x) \equiv \alpha x^n$$

On carrying out the integration in (4.3) and (4.4) we obtain

$$x = A\tau - \frac{Aa\tau^3}{6} - \frac{qA}{2}(\cos 2\tau + 1)\tau + \frac{qA}{2}\sin 2\tau - \frac{\alpha A^n \tau^{n+2}}{(n+1)(n+2)};\quad a\frac{\pi^2}{8} + q + \frac{\alpha}{n+1}\left(\frac{\pi}{2}\right)^{n+1} A^{n-1} = 1$$

For $\alpha \neq 0$, the last relation is considered as an equation in A; if $\alpha = 0$ (the Mathieu equation) we obtain the approximate expression connecting the eigenvalue a and the parameter q.

5. In combination with averaging methods, the approximate solutions of the type described make it possible, comparatively simply, i.e. without recourse to complex special functions, to consider systems close to conserviatve in cases where the quasiharmonic approximation of the solution proves to be ineffective.

We shall illustrate this by using the equation

$$x^{\cdot\cdot} + g(x, x^{\cdot}) + \operatorname{sh} x = 0;\ g(x, x^{\cdot}) \equiv (bx^2 - 1)\, x^{\cdot}$$

as an example.

Considering the asymptotic solutions for the case $g(x, x^{\cdot}) \equiv 0$ (see 1.17),

$$x = AP(\varphi),\ x^{\cdot} = e^{A/2}P'(\varphi);\ A \to \infty$$

as the formulae for changing to the new variables $A(t), \varphi(t)$ for $g(x, x^{\cdot}) \not\equiv 0$, we arrive at the equations

$$A^{\cdot} = -2e^{-A/2} g(AP, e^{A/2}P')\, P',\quad \varphi^{\cdot} = \frac{e^{A/2}}{A} + \frac{2e^{-A/2}}{A} g(AP, e^{A/2}P')\, P$$

We will restrict ourselves to the shortened system (see /4/)

$$A = 2\left(1 - \frac{b}{3}A^2\right),\quad \varphi^{\cdot} = \frac{e^{A/2}}{A}$$

The solution of the first equation is

$$A = \sqrt{\frac{3}{b}} \begin{Bmatrix}\text{th}\\ \text{ctb}\end{Bmatrix} \left[2\sqrt{\frac{b}{3}}(t + t_0)\right],\quad t_0 = \text{const}$$

As $t \to \infty$, we obtain a selfoscillating mode with 'amplitude' $A = \sqrt{3/b}$. Thus, the approximation of the solution of a sawtooth function and the averaging procedure have been substantiated for small values of the parameter b.

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A NEW CLASS OF EXACT SOLUTIONS WITH SHOCK WAVES IN GAS DYNAMICS *

S.A. POSLAVSKII

New exact solutions of the equation of one-dimensional gas dynamics with strong shock waves propagating in a moving medium are obtained. The gas flow behind a discontinuity is described by a solution with uniform deformation (see /1, 2/). Solutions of the explosion problem without a counterpressure in a uniformly expanding (or compressing) gas with an arbitrary adiabatic exponent and a non-uniform initial density distribution are constructed, as well as of the problem of cavity collapse in a dust cloud with the formation of a shock wave.

The solution (see /1, 2/) was joined with the shock and detonation waves propagating in a quiescent gas in /3 - 6/. The problem of joining, by the use of the shock wave, of a solution for a moving selfgravitating medium with zero pressure, and the problem of selfsimilar solutions were discussed in /7/. An exact solution of the problem of a strong explosion in a uniformly expanding (or compressing) gas with a special adiabatic exponent equal to 5/3 was obtained in /8/.

1. The exact particular solution of a system of equations, which describes the one-dimensional adiabatic motion of an ideal gas, found by L.I. Sedov, /1, 2/, can be represented by the formulae

$$r = R(t)\xi,\ dR = \pm[2\varepsilon\lambda^{-1}(R^{-\lambda}+A)]^{1/2}dt,\ \lambda=\nu(\gamma-1) \tag{1.1}$$

$$v = R^{-1}R^{\cdot}r,\quad p=\frac{p_0(\xi)}{R^{\nu\gamma}},\quad \rho=\frac{\rho_0(\xi)}{R^{\nu}} \tag{1.2}$$

$$p_0(\xi)=p_0(\xi_0)\exp\int_{\xi_0}^{\xi}\frac{\varepsilon\xi\,d\xi}{G(\xi)},\quad \rho_0(\xi)=\frac{p_0(\xi)}{G(\xi)}$$

(the dot denotes a derivative with respect to time t).

Here r and ξ are the Euler and Lagrange coordinates, v is the velocity, p is the pressure, ρ is the density, γ denotes the adiabatic exponent $(\gamma>1)$, $\nu=1,2,3$ for motions with plane, cylindrical and spherical waves respectively, A, ε, ξ_0, $p_0(\xi_0)$ are arbitrary constants, and $\xi_0>0$, $p_0(\xi_0)>0$, $G(\xi)$ is an arbitrary function. By correctly selecting the Lagrangian coordinates we can have $\varepsilon=\pm1$.

Let us consider the problem of joining the solution of (1.1), (1.2) with a shock wave which propagates in a gas with zero pressure (in a dust medium). We write the conditions at the discontinuity denoting the quantities in front of the shock wave by the index 1, taking into account that $p_1=0$ and using relations (1.1) and (1.2):

$$\rho_1=\frac{\gamma-1}{\gamma+1}\frac{p_0(\xi_0)}{R^{\nu}G(\xi_*)}\exp\int_{\xi_0}^{\xi_*}\frac{\varepsilon\xi\,d\xi}{G(\xi)} \tag{1.3}$$

$$\xi_*^{\cdot}=\left[\frac{\gamma-1}{2}R^{-\lambda-2}G(\xi_*)\right]^{1/2},\quad G(\xi_*)=\frac{\gamma-1}{2}R^{\lambda}(R^{\cdot}\xi_*-v_1)^2 \tag{1.4}$$

($\xi_*=r_*/R$ is a Lagrange coordinate of gas particles which occur at the shock wave front, and r_* is the shock wave radius).

The motion of the dust medium before a discontinuity is determined by the relation

*Prikl.Matem.Mekhan.,49,5,752-757,1985